Virasoro constraints in sheaf theory

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Vertex algebras

Joint work w/ A. Bojko, M. Moreira

Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$.

I. Virasoro constraints

II. Vertex algebras

III. Wall-crossing formulas

I. Virasoro constraints

$$\left\{ \begin{array}{lll} X & \text{smooth projective variety} / \mathbb{C} \\ \alpha & \in H^*(X, \mathbb{Q}) \\ H & \text{ample line bundle} \end{array} \right.$$

$*$ $M_X^{H\text{-ss}}(\alpha) = M$: moduli of H-semistable sheaves F

s.t. $ch(F) = \alpha$

very sensitive to X, α

"less sensitive" to H $\Longrightarrow$ wall-crossing formulas

Assumptions ① $\nexists$ strictly H-semistable sheaf in M

② $Ext^i(F,F) = 0$ $\forall F \in M$, $\forall i \geq 3$

③ $\exists$ universal sheaf $\mathbb{F}$ on $M \times X$

① + ② $\Longrightarrow$ virtual fundamental class

$$[M]^{vir} \in A_{vdim}(M) \xrightarrow{cl} H_{2vdim}(M, \mathbb{Z}) \dashrightarrow H_{2vdim}(\mathcal{M}_X^{rig \, \alpha})$$

$1 - \chi(F, F)$

③ $\Longrightarrow$ natural cohomology classes $\xi_{\mathbb{F}} : \mathbb{D}^X \longrightarrow H^*(M, \mathbb{C})$

Def. Descendent algebra $\mathbb{D}^X := \left\langle \mathrm{ch}_i^H(\gamma) \;\middle|\; i \geq 0,\ \gamma \in H^*(X,\mathbb{C}) \right\rangle$

formal symbols

$$* \quad \mathrm{ch}_i^H(\lambda_1\gamma_1 + \lambda_2\gamma_2) = \lambda_1 \cdot \mathrm{ch}_i^H(\gamma_1) + \lambda_2 \cdot \mathrm{ch}_i^H(\gamma_2)$$

Realization homomorphism $\xi_{\mathbb{F}} : \mathbb{D}^X \longrightarrow H^*(M,\mathbb{C})$

$$\begin{tikzcd} & \mathbb{F} \arrow[d] & \\ & M \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M & & X \end{tikzcd}$$

$$\mathrm{ch}_i^H(\gamma) \longmapsto \pi_{1*}\left(\mathrm{ch}\,\mathbb{F} \cdot \pi_2^*\gamma\right)$$

$i + \dim X - p$ if $\gamma \in H^{p,*}(X,\mathbb{C})$.

$$\Longrightarrow \xi_{\mathbb{F}}(\mathrm{ch}_i^H(\gamma)) \in \text{"}H^{i,*}(M,\mathbb{C})\text{"}$$

Virasoro operators $\mathsf{L}_n = R_n + T_n \circlearrowright \mathbb{D}^X$

* R_n : derivation s.t.

$$R_n\left(\mathrm{ch}_i^H(\gamma)\right) = i(i+1)\cdots(i+n)\,\mathrm{ch}_{i+n}^H(\gamma)$$

* T_n : multiplication by

$$T_n = \sum_{i+j=n} i!\,j! \sum_s (-1)^{\dim X - p_s^L}\,\mathrm{ch}_i^H(\gamma_s^L)\,\mathrm{ch}_j^H(\gamma_s^R)$$

where $\Delta_* \mathrm{td}(X) = \sum_s \gamma_s^L \otimes \gamma_s^R$

Indeed, $[L_n, L_m] = (m-n)\,L_{n+m}$.

Wrong guess $\displaystyle\int_{[M]^{vir}} \xi_{\mathbb{F}}(L_n(D)) \neq 0, \quad \forall n \geq -1 \quad \forall D \in \mathbb{D}^X.$

1) introduce "correction operator"

Conjecture ([Moreira-Oblomkov-Okounkov-Pandharipande], [M], [van Bree])

$$\int_{[M]^{vir}} \xi_{\mathbb{F}}\left(\left(L_n + \tfrac{1}{r} S_n\right)(D)\right) = 0, \quad \forall n \geq -1, \forall D \in \mathbb{D}^X.$$

* [MOOP, M] : $M = PT_n(X,\beta)$, $S^{[n]}$
* [vB] : $M = M_S^{H\text{-ss}}(r, c_1, c_2)$ where $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$

2) combine $\{L_n\}_{n \geq -1}$ to define

$$\mathbf{L_{wt_0}} := \sum_{n=-1}^{\infty} \frac{(-1)^n}{(n+1)!} L_n \circ R_{-1}^{n+1} : \mathbb{D}^X \longrightarrow \underline{\mathbb{D}^X_{\mathbf{wt_0}}}$$

$:= \ker(R_{-1})$

* $\xi = \xi_{\mathbb{F}} : \mathbb{D}^X_{wt_0} \longrightarrow H^*(M, \mathbb{C})$

Conjecture (BLM) $\displaystyle\int_{[M]^{vir}} L_{wt_0}(D) = 0, \quad \forall D \in \mathbb{D}^X.$

Proposition (BLM) 1) $\Longleftrightarrow$ 2) for old cases.

New cases : $M = M_C(r,d)$, $M_S^{H\text{-ss}}(0, \beta, \chi)$, $M_X^{H\text{-ss}}(r, c_1, c_2, c_3)$

Fano 3-fold

Variation for pairs

$P_X^\sigma(V,\alpha) = P$: moduli of "σ-semistable" pairs $[V \xrightarrow{\varphi} F]$ s.t. $ch(F)=\alpha$.

<u>Assumptions</u> ① $\nexists$ strictly σ-semistable pair in P

② $Ext^i([V\to F],F)=0$ unless $i=0,1$.

③ $\exists!$ universal pair $[\pi_2^* V \xrightarrow{\psi} \mathbb{F}]$ on $P\times X$.

$$\Rightarrow [P]^{vir},\ \xi_{(\pi_2^*V,\mathbb{F})} : \mathbb{D}^X \otimes \mathbb{D}^X \longrightarrow H^*(P,\mathbb{C})$$

L_n^{pair}

<u>Conjecture</u> (BLM) $\displaystyle\int_{[P]^{vir}} \xi_{(\pi_2^*V,\mathbb{F})}\left(L_n^{pair}(D)\right) = 0 \quad \forall n\geq 0, \forall D$.

e.g. $C^{[n]}$, $S_\beta^{[0,m]}$, $Quot_S(V,\beta,n)$, $P_S^{t-ss}(\alpha)$

$t\in(0,\infty)$

$[\mathcal{O}_S\to F]$ is μ_t-semistable if

* $0\neq {}^\forall G\subsetneq F$, $\mu(G)\leq \mu(F)+\frac{t}{r(F)}$

* $s:\mathcal{O}_S\to {}^\forall G\subsetneq F$, $\mu(G)+\frac{t}{r(G)}\leq \mu(F)+\frac{t}{r(F)}$

II. Vertex algebras

$$\left(V_\bullet \,,\ |0\rangle \,,\ T \,,\ Y(-,z) \right)$$

$\mathbb{Z}$-graded vector space

$$V_\bullet = \bigoplus_{i\in\mathbb{Z}} V_i$$

vacuum vector

$$|0\rangle \in V_0$$

translation map

$$T : V_\bullet \to V_{\bullet+2}$$

state-field correspondence

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(V_\bullet)[[z,z^{-1}]]$$

Axioms for these data are related to 2d conformal field theory.

e.g. Locality : $\forall\, a,b \in V_\bullet \quad \exists N$ s.t. $(z-w)^N \left[Y(a,z), Y(b,w) \right] = 0$.

Example (free boson) $\ast$ $\mathfrak{h}$ $\mathbb{C}$-vector space

$\ast$ $\langle\, ,\, \rangle$ symmetric non-degenerate form

1) $V_\bullet := \mathrm{Sym}\left[\bigoplus_{k>0} \mathfrak{h}\, t^{-k} \right]$ where $\deg(t) = -2$.

Write $a \cdot t^{-k} := a_{-k}$.

2) $|0\rangle = 1$

3) T = derivation s.t. $T(a_{-k}) = k \cdot a_{-k-1}$

4) $\forall\, a \in \mathfrak{h}$, $Y(a_{-1}, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(V_\bullet)[[z,z^{-1}]]$

$\ast$ $\left[a_{(n)}, b_{(m)} \right] = \delta_{n+m,0}\ n \cdot \langle a,b\rangle \quad \forall\, n,m \in \mathbb{Z}$

$\ast$ $a_{(-k-1)} = \cdot\, a_{-k-1}$, $a_{(k)}|0\rangle = 0 \quad \forall\, k \geq 0$

Virasoro constraints in sheaf theory $\longleftrightarrow$ Vertex algebras

* In many examples, $\exists$ Virasoro algebra $\circlearrowright V_\bullet$

Lie algebra spanned by $\{L_n\}_{n\in\mathbb{Z}}$ & c.

* Recently, D. Joyce introduced sheaf theoretic vertex algebra.

$$(\mathcal{M}_X, 0, \Sigma, \Psi, \Theta) \rightsquigarrow V_\bullet^{Joyce} = H_*(\mathcal{M}_X, \mathbb{C})$$

Crucially, $[M]^{vir} \in H_*(\mathcal{M}_X^{rig}, \mathbb{C}) \simeq \underbrace{\check{V}_\bullet^{Joyce} := V_{\bullet+2}/T(V_\bullet)}$

$\exists$ Lie bracket operation !

<u>Def</u>. $\mathbf{w} \in V_\bullet$ is a conformal element if

$$Y(\mathbf{w}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfies 1) $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0} \cdot \frac{n^3-n}{12} \cdot c$

2) $L_0 \circlearrowright V_\bullet$ diagonalizable.

<u>Example</u>. $V_\bullet = \mathrm{Sym}\left[\bigoplus_{k>0} \mathfrak{h}t^{-k}\right]$ $\quad c = \dim(\mathfrak{h})$: central charge

$\quad\;\cup$

$$\mathbf{w} := \frac{1}{2}\sum_{a\in B} \hat{a}_{-1} \cdot a_{-1}$$

Assume X is either curve or surface w/ $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$

Thm (BLM) There is a natural conformal element

$$\omega \in V_\bullet^{pair} = H_*(\mathcal{M}_X \times \mathcal{M}_X, \mathbb{C})$$

s.t.

$$H^*(\mathcal{M}_\alpha \times \mathcal{M}_\beta) \otimes H_*(\mathcal{M}_\alpha \times \mathcal{M}_\beta) \longrightarrow \mathbb{C}$$

L_n^{pair} : Virasoro operator on $\mathbb{D}^X \otimes \mathbb{D}^X$ $\longleftrightarrow$ dual L_n^{pair} : Virasoro operator from $\underline{\underline{\omega}}$

Remark We use the computation of $H_*(\mathcal{M}_\alpha)$ by [Gross].

In particular, $\mathbb{D}^X \otimes \mathbb{D}^X \xrightarrow{\xi} H^*(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$.

Corollary ① M satisfies Virasoro constraints

$$\Longleftrightarrow \left[[M]^{vir}, \omega \right] = 0 \in V_\bullet^{pair}$$

$$\Longleftrightarrow [M]^{vir} \in \check{P}_0 \subset (\check{V}_\bullet^{pair}, [\,,\,])$$

② P satisfies Virasoro constraints

$$\Longleftrightarrow L_n^{pair}([P]^{vir}) = 0 \in V_\bullet^{pair}, \quad \forall n \geq 0$$

$$\Longleftrightarrow [P]^{vir} \in P_0 \subset V_\bullet^{pair}$$

$\check{P}_0$: Lie subalgebra

P_0 : Lie alg rep'n

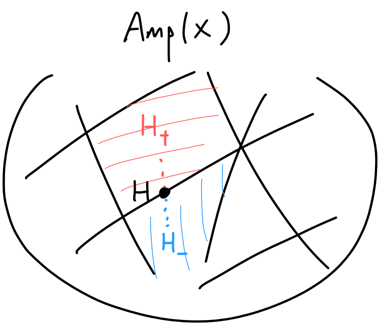
※ $\check{P}_0$, P_0 : space of primary/physical states.

III. Wall-crossing formulas

Thm (Joyce) Wall-crossing formulas for $[M_X^{H\text{-ss}}(\alpha)]^{inv} \in (\check{V}_\bullet^{Joyce}, [,])$

are explicitly written in terms of $[,]$.

Example (Simple wall-crossing)



$F \in M_X^{H_+\text{-ss}}(\alpha) \setminus M_X^{H_-\text{-ss}}(\alpha)$

iff $0 \to F_1 \to F \to F_2 \to 0$, $F_i \in M_X^{H\text{-ss}}(\alpha_i)$ irreducible

* similarly for other complement with F_1, F_2 swapped

$$\Rightarrow \quad [M_X^{H_+\text{-ss}}(\alpha)]^{vir} - [M_X^{H_-\text{-ss}}(\alpha)]^{vir} = \Big[[M_X^{H\text{-ss}}(\alpha_1)]^{vir}, [M_X^{H\text{-ss}}(\alpha_2)]^{vir}\Big]$$

Theorem (Bojko-L-Moreira)

Virasoro constraints hold for moduli of torsion-free sheaves on any curves and surfaces with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$.

key ideas of proof

* rank = 1 case : Virasoro constraints for $S^{[n]}$ [Moreira]

* wall-crossing [Joyce]

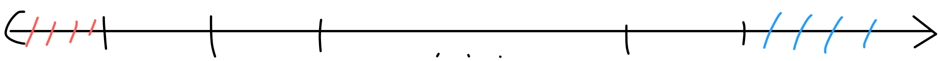
  1) $P_S^{\infty}(\alpha) \longleftrightarrow P_S^{0t}(\alpha)$, $\quad t \in (0,\infty)$

  2) $\gamma_\alpha = \chi(\alpha)\cdot\left[M_X^{H\text{-}ss}(\alpha)\right]^{inv} + (\text{low rank } \alpha\text{'s})$

* Relation to Virasoro constraints [BLM]

  1) wall-crossing compatibility ($\check{P}_0, P_0$: Lie algebra, rep'n)

  2) projective bundle compatibility.



② $P_S^{0t}(\alpha) \xleftrightarrow[t \in \mathbb{R}_{>0}]{}$ ① $P_S^{\infty}(\alpha) = \begin{cases} \emptyset & \text{if } rk\,\alpha > 1 \\ S_\beta^{[0,m]} & \text{if } rk\,\alpha = 1 \end{cases}$

$\Big\Downarrow$ projective bundle compatibility

③ $\gamma_\alpha \overset{\text{def}}{:=} f_*\left(c_{top}(T_f) \cap \left[P_X^{0t}(\alpha)\right]^{vir} \right)$

$= \underbrace{\chi(\alpha)}_{>0} \underbrace{\left[M_X^{H\text{-}ss}(\alpha)\right]^{inv}}_{④} + (\text{lower rank})$